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Some Results on Cone Metric Spaces Introduced by Jungck Multistep Iterative Scheme

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ABSTRACT

In this paper, we have introduced some common fixed point theorems, which satisfy the Jungck iterative process for compatible mapping in cone metric space.

***Keywords**: Common fixed point, weakly compatible mapping, Cone metric space, Jungck multistep iteration.*

I. INTRODUCTION

An abstract metric space was first introduced and studied by the French mathematician Frechet [22] in1906. Many researchers have been generalized the concept of metric space as cone metric space, semi metric space and quasi metric space etc, along with the generalization of contraction mappings with applications [2, 3, 4, 5]. The concept of common fixed point theorems have been introduced by Jungck [6, 7, 8, 9], which is generalized the Banach contraction principle [27]. Some interesting work related to generalization of the contraction mapping and metric space can be seen in [11, 14-18, 31, 32] and its references. Banach contraction principle is a basic fundamental theorem for fixed point theory. Further, Jungck [7, 8, 10] introduced weakly compatible maps in metric space.

The concept of cone metric space was introduced by Huang and Zhang [13], which is generalization of metric space in order to replace the real number with Banach space [25].

The Jungck multistep iteration scheme are generalization of Jungck-Mann, Jungck-Noor, Jungck-Ishikawa iteration in cone Banach space, firstly, who introduced by Olalere and Akewe [26].

Banach valued metric space was considered by Rzepecki [1], Lin [28] and lately by Huang and Zhang [13]. Basically for nonempty set X, the definition of metric d: $X \times X \rightarrow R^{+} = [0, \infty)$ is replaced by a new metric, simply by an ordered Banach spaces E, d: $X \times X \rightarrow E$ such that space are called cone metric spaces (in short CMSs)

In this paper, we obtain some points of coincidence and common fixed points for two self-mappings satisfying generalized Jungck iteration process (i.e. Jungck multiple iteration) in cone metric space.

Definition 1.1 [19, 20] we define the cone metric spaces and their convergence by [13] Huang and Jungck.

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Let E be a real Banach space and P be a subset of E. The subset P is called Cone if it has the following properties

(i) P is nonempty, closed and $P \neq 0$.

(ii) $0 \preccurlyeq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$

(iii) $P \cap (-P) = 0$.

For a given cone $P \subseteq E$, we can define partial ordering $\preccurlyeq$ on E with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We will write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$, while $x \leq y$ will stand for $y - x \in intP$, where *int*P denotes the interior of P. The cone P is called normal if there exist a constant $M \geq 0$ such that for all $x, y \in E$

$0 \leq x \leq y \Rightarrow \| x \| \leq M \| y \|$.

The least positive number $M \geq 0$ satisfying the above inequality is called normal constant of P.

The cone P is a non-normal cone if and only if there exist a sequence $u_n, v_n \in P$ such that $0 \leq u_n \leq u_n + v_n, u_n + v_n \to 0$.

Example 1.1 [12] Let $E = C^1[0,1]$ with $\| x \| = \| x \|_\infty + \| x' \|_\infty$ on $P = \{x \in E, x(t) \geq 0$ on $[0,1]\}$. Clearly, this cone is not normal. To see it, consider

$$x_n = \frac{1 - cos3nt}{3n + 2} \; and \; y_n = \frac{1 + cos3nt}{3n + 2}.$$

Then we have $\| x_n \| = \| y_n \| = 1$ and $x_n + y_n = \frac{2}{3n+2} \to 0$.

The cone P is called regular, if every increasing (or decreasing) and bounded above (or below) sequence is convergent in E. If x_n is a sequence such that $x_1 \leq x_2 \leq x_3 \ldots\ldots \leq x_n \ldots\ldots\ldots \leq y$ (or $y \leq \ldots \leq x_n \leq \ldots \leq x_3 \leq x_2 \leq x_1$) for some $y \in E$, then there is a $x \in X$ such that $\| x_n - x \| \to 0, n \to \infty$.

Equivalently the cone P is regular if and only if increasing (respectively decreasing) sequence which is bounded from above (respectively below) is convergent. It is well known that a regular cone is a normal cone.

Definition 1.2 [19, 20] Let X is a nonempty set and E is a real Banach space. Suppose that mapping $d: X \times X \to E$ satisfy the following

(i) $0 \leq d(x, y) for\ all\ x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and x ∈ X , if for every c ∈ E, with $0 \ll c$ there is $n_0 \in N$ such that for all n ≥ n_0, d(x_n, x) ≪ c , then x_n is said to be convergent , $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n$ = x or $x_n \to x$ as n→ ∞. If for every c ∈ E with $0 \ll c$. There is $n_0 \in N$ such that for all n, m ≥ n_0 , d(x_n, x_m) ≪ c, then x_n is called a Cauchy sequence in X. If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

Definition 1.3 [19, 20] A point y ∈ X is called point of coincidence of a family f_i, i ∈ I^+ of self-mappings on X , if there exist a point x ∈ X such that f_i x = y for all i ∈ I^+ , x is called coincidence point of mapping $\{f_i\}_{i=1}^{\infty}$.

Definition 1.4 [23, 24, 26, 30] Let (X, d) be a cone Banach space and f, g: X × X → E be two mappings such that f(X) ⊆ g(X). For any x_0 ∈ X, the sequence $\{gx_n\}_{n=1}^{\infty}$ is defined by Jungck iterative scheme as follows

$$gx_{n+1} = fx_n,\ n \geq 0 \tag{1.1}$$

In the similar way [33], for any $x_0 \in X$ the Jungck Mann iterative scheme $\{gx_n\}_{n=1}^{\infty}$ is defined a

$$gx_{n+1} = (1 - \alpha_n)\, gx_n + \alpha_n\, fx_n\ ,\ n \geq 0\ . \tag{1.2}$$

Where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. It is also known as Jungck one step iteration scheme.

For any $x_0 \in X$, the Jungck Ishikawa iteration scheme $\{gx_n\}_{n=1}^{\infty}$ is generalized to Ishikawa iteration [29] as follows

$$gx_{n+1} = (1 - \alpha_n)\, gx_n + \alpha_n\, fy_n\ ,$$

$$gy_n = (1 - \beta_n)\, gx_n + \beta_n\, fx_n\ ,\ n \geq 0\ . \tag{1.3}$$

Where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequence in [0, 1]. It is also known as Jungck two step iterative scheme.

The generalization of Noor iterative scheme [21] is known a Jungck Noor iterative scheme and is defined as, for any $x_0 \in X$ the sequence $\{gx_n\}_{n=1}^{\infty}$ is expressed as

$$gx_{n+1} = (1 - \alpha_n)\, gx_n + \alpha_n\, fy_n\ ,$$

$$gy_n = (1 - \beta_n)\, gx_n + \beta_n\, fz_n\ ,$$

$$gz_n = (1 - \gamma_n)\, gx_n + \gamma_n\, fx_n\ ,\ n \geq 0. \tag{1.4}$$

Where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequence in [0, 1]. It is also known as Jungck three step iterative scheme.

In the continuation of Jungck-Mann, Ishikawa and Noor iteration, Olaleru and Akewe [26] defined the multistep iteration mapping known as Jungck multistep iterative scheme, which is defined in cone Banach space as follows.

Let $x_0 \in X$ the Jungck multistep iterative scheme for the sequence $\{gx_n\}_{n=1}^{\infty}$ is defined by

$$gx_{n+1} = (1 - \alpha_n)\, gx_n + \alpha_n\, fy_n^1,$$

$$gy_n^i = (1 - \beta_n^i)\, gx_n + \beta_n^i\, fz_n^{i+1},\ \mathrm{I} = 1, 2, 3, \ldots\ldots k\text{-}2.$$

$$gy_n^{k-1} = (1 - \beta_n^{k-1})\, gx_n + \beta_n^{k-1}\, fx_n,\ k \geq 2,\ n \geq 0. \tag{1.5}$$

Where $\{\alpha_n\}_{n=1,}^{\infty}$ $\{\beta_n^i\}_{n=1}^{\infty}$ i = 1, 2 3, ….., K-1 are real sequences in [0,1].

II. MAIN RESULTS

Theorem 2.1 *Let (X, d) be a cone metric space and P is a cone in E. Consider two mappings f, g: X → E, which have coincidence point z ∈ X (i.e fz = gz = p) with f(X) ⊆ g(X) and satisfy*

$$d(fx, fy) \leq \left[\frac{\delta\, d(gx, gy) + \psi\,(d(gx, fx))}{1 + M\, d(gx, fx)}\right], 0 \leq \delta < 1, M \geq 0. \tag{2.1}$$

Where ψ is a monotonic increasing and continuous function such that $\psi(0) = 0$, then Jungck multiple iteration $\{gx_n\}_{n=1}^{\infty}$ converges to p. Further, if f, g commutes at p (i.e. f and g are weakly compatible), then p is the unique common fixed point of f and g.

Proof: From (1.5), we discuss

$$d(gx_{n+1}, p) = d((1 - \alpha_n)gx_n + \alpha_n f y_n^1, p)$$

$$\leq (1 - \alpha_n)\, d(gx_n, p) + \alpha_n\, d(fy_n^1, p)$$

$$= (1 - \alpha_n)\, d(gx_n, p) + \alpha_n\, d(fy_n^1, fz)$$

$$= (1 - \alpha_n)\, d(gx_n, p) + \alpha_n\, d(fz, fy_n^1)$$

from (2.1), we have

$$= (1 - \alpha_n) d(gx_n, p) + \alpha_n \left[\frac{\delta\, d(gz, gy_n^1) + \psi\,(d(gz, fz))}{1 + M\, d(gz, fz)}\right]$$

$$= (1 - \alpha_n)\, d\,(gx_n, p) + \alpha_n\, \delta\, d(gz, gy_n^1), \qquad (fz = gz = p). \tag{2.2}$$

From (1.5) and (2.1), we have

$$d(gy_n^1, p) \leq d((1 - \beta_n^1)gx_n + \beta_n^1 f y_n^2, p)$$

$$= (1 - \beta_n^1)\, d(gx_n,\, p) \;+\; \beta_n^1\, d(fy_n^2,\, p)$$

$$= (1 - \beta_n^1)\, d(gx_n,\, p) \;+\; \beta_n^1\, d(fz,\, fy_n^2)]$$

$$= \;(1 - \beta_n^1)\, d(gx_n,\;\, p) + \beta_n^1 \left[\frac{\delta\, d(gz,\;\, gy_n^2) + \psi\, (d(gz,\;\, fz))}{1 + M\, d(gz,\, fz)}\right]$$

$$= (1 - \beta_n^1)\, d(gx_n,\, p) \;+\; \beta_n^1\, \delta\, d(gz,\, gy_n^2)\;. \tag{2.3}$$

Putting (2.3) in (2.2), we obtain

$$\mathrm{d}(\mathrm{g}x_{n+1},\, p) \;\le\; (1 - \alpha_n)\, \mathrm{d}\, (\mathrm{g}x_n, p) + \delta\, \alpha_n\, (1 - \beta_n^1)\, d(gx_n,\, p) \;+\; \beta_n^1\, \delta^2\, \alpha_n\, d(gz,\, gy_n^2)$$

$$= (1 - \alpha_n(1 - \delta) - \delta\, \alpha_n\, \beta_n^1\,)\, d(gx_n,\, p) \;+\; \beta_n^1\, \delta^2\, \alpha_n\, d(gz,\, gy_n^2)$$

$$= (1 - \alpha_n(1 - \delta) - \delta\, \alpha_n\, \beta_n^1\,)\, d(gx_n,\, p) \;+\; \beta_n^1\, \delta^2\, \alpha_n\, d(p,\, gy_n^2). \tag{2.4}$$

Similarly,

$$d(\, gy_n^2,\, p) = \; d((1 - \beta_n^2\,)\, gx_n + \beta_n^2\, fx_n^3\,, \mathrm{p})$$

$$\le\; (1 - \beta_n^2\,) d\, (\, gx_n,\, p) + \beta_n^2\;\, d(\, fx_n^3\,, \mathrm{p})$$

$$=\; (1 - \beta_n^2\,) d\, (\, gx_n,\, p) + \beta_n^2\;\, d(\, fz,\, fx_n^3\,)$$

$$\le (1 - \beta_n^2)\, d(gx_n,\;\, p) + \beta_n^2 \left[\frac{\delta\, d\left(gz,\, gy_n^3\right) + \psi\, (d\, (gz,\, fz))}{1 + M\;\, d\, (gz,\;\, fz)}\right]$$

$$=\; (1 - \beta_n^2\,) d\, (\, gx_n,\, p) + \delta\, \beta_n^2\;\, d(\, gz,\, gy_n^3\,). \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$\mathrm{d}(\mathrm{g}x_{n+1},\, p) \le (\, 1 - \;\; (1 \; - \; \delta\,)\, \alpha_n \;\; - \; \delta\, \alpha_n\, \beta_n^1\,)\;\, d(gx_n,\, p)$$

$$+\, \delta^2 \alpha_n \beta_n^1\, (\, 1 - \; \beta_n^2) d(gx_n,\, p) + \; \delta^3 \alpha_n \beta_n^1 \beta_n^2\, d(p,\, gy_n^3)$$

$$= (1 - (1 - \; \delta)\alpha_n \; - (1 - \; \delta)\delta\alpha_n\beta_n^1 - \; \delta^2\alpha_n\beta_n^1\beta_n^2\,)\, d(gx_n\,,\;\, p) \;+\; \delta^3\alpha_n\beta_n^2\;\, d(p,\, gy_n^3). \tag{2.6}$$

Similar process as in (2.3) and (2.5) , we have

$$d\big(gy_n^3,\, p\big) \;\le \big(1 - \;\, \beta_n^3\big)\, d(gx_n,\, p) + \; \delta\beta_n^3\, d(y_n^4,\, p). \tag{2.7}$$

From (2.6) and (2.7) and proceed them, we have

$$d\, (gx_{n+1},\, p) \;\le (\, 1 - \;\; (1 \, - \, \delta\,)\, \alpha_n \;\; - \; \delta^{k-2}\, \alpha_n\;\, \beta_n^1\, \beta_n^2 \beta_n^3 \ldots\ldots \beta_n^{k-2})\;\, d(gx_n,\, p)$$

$$+\;\, \delta^{k-1} \alpha_n \beta_n^1 \beta_n^2 \beta_n^3 \ldots\ldots\ldots \beta_n^{k-2} d(p,\, gy_n^{k-1})$$

$$\le (1 - (1 - \delta)\alpha_n - \delta^{k-2}\alpha_n \beta_n^1 \beta_n^2\beta_n^3.....\beta_n^{k-2})\ d(gx_n, p)$$

$$+ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}[(1 - \beta_n^{k-2})d(p, gx_n) + \beta_n^{k-1} d(fz, fx_n)]$$

$$\le (1 - (1 - \delta)\alpha_n - \delta^{k-2}\alpha_n \beta_n^1 \beta_n^2\beta_n^3.....\beta_n^{k-2})\ d(gx_n, p)$$

$$+ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}[(1 - \beta_n^{k-2})d(p, gx_n) + \delta\beta_n^{k-1} d(p, gx_n)]$$

$$\le (1 - (1 - \delta)\alpha_n - \delta^{k-2}\alpha_n \beta_n^1 \beta_n^2\beta_n^3.....\beta_n^{k-2}) + (\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}\ d(gx_n, p)$$

$$\le (1 - (1 - \delta)\alpha_n)\,\mathrm{d}(gx_n, p)$$

$$\le (1 - (1 - \delta))\,\mathrm{d}(gx_n, p)$$

$$\Rightarrow \mathrm{d}(gx_{n+1}, p) \le \mathrm{d}(gx_n, p). \qquad (2.8)$$

Hence $gx_n \to p$, since $1 - \delta < 1$ for all n.

Now, we show that p is unique. Suppose there exist another point of coincidence is p^*, then there is an $z^* \in X$, such that $fz^* = gz^* = p^*$.

Now,

$$d(p, p^*) = d(fz, fz^*) \le \frac{\delta\, d(gz, gz^*) + \psi(d(gz, fz))}{1 + M\, d(gz, fz)} = \delta d(gz, gz^*) = \delta\, d(p, p^*).$$

$$(1 - \delta)d(p, p^*) \le 0, \quad [\because (1 - \delta) < 1]$$

$$\Rightarrow d(p, p^*) \le 0 \Rightarrow d(p, p^*) = 0.$$

So p = p^* (i.e. p is unique).

Since $fz = gz = p$, then $fgz = fp$ and $gfz = gp$ but $fgz = gfz$, so $fz = gz$. i.e. $fz = gz = p$ or $fp = gp = p$. So z is unique common fixed point of f *and* g.

Theorem 2.2 *Let (X, d) be a cone metric space and P is a cone in E. Consider two mappings $f, g: X \to E$, which is weakly compatible at coincidence point p with $f(X) \subseteq g(X)$ and satisfy*

$$d(fx, fy) \le \delta d(gx, gy) + L\,(d(gx, fx)),\ 0 \le \delta < 1, L \ge 0. \qquad (2.9)$$

Then the Jungck multistep iteration $\{gx_n\}_{n=1}^{\infty}$ converges to p and f,g have unique common fixed point p.

Proof: From (1.5) and (2.9) with the fact that $fz = gz = p$, we have

$$d(gx_{n+1}, p) = d((1 - \alpha_n)gx_n + \alpha_n f y_n^1, p)$$
$$\le (1 - \alpha_n)\, d(gx_n, p) + \alpha_n\, d(fy_n^1, p)$$

$$= (1- \alpha_n)\, d(gx_{n,} p) + \alpha_n\, d(fy_n^1 , fz)$$

$$= (1- \alpha_n)\, d(gx_{n,} p) + \alpha_n\, d(fz , fy_n^1)$$
$$= (1- \alpha_n)\, d(gx_{n,} p) + \alpha_n\, [\, \delta\, d(gz , gy_n^1) + L\, (d(gz, fz))]$$

$$= (1- \alpha_n)\, d\big(gx_{n,} p\big) + \alpha_n\ \delta\, d(gz , gy_n^1) \quad (fz = gz = p). \tag{2.10}$$

Again from (1.5) and (2.9), we have

$$d(gy_n^1, p) \leq d((1- \beta_n^1)gx_n + \beta_n^1 fy_n^2, p)$$

$$= (1- \beta_n^1)\, d(\, gx_n, p) + \beta_n^1\ d(fy_n^2, p)$$

$$= (1- \beta_n^1)\, d(\, gx_n, p) + \beta_n^1\ d(fz , fy_n^2)$$

$$= (1- \beta_n^1)\, d(\, gx_n, p) + \beta_n^1\ [\delta\, d(gz , gy_n^2) + L(d(gz, fz))]$$

$$= (1- \beta_n^1)\, d(\, gx_n, p) + \beta_n^1\ \delta\, d(gz , gy_n^2) . \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$d(gx_{n+1}, p) \leq (1- \alpha_n) d(gx_n, p) + \delta\, \alpha_n\, (1- \beta_n^1)\, d(gx_n, p) + \delta^2 \beta_n^1 \alpha_n\, d(p, gy_n^2)$$

$$= (1- \alpha_n(1- \delta) - \delta\, \alpha_n\ \beta_n^1)\, d(gx_n, p) + \delta^2 \beta_n^1 \alpha_n\, d(p, gy_n^2) . \tag{2.12}$$

Similarly, we have

$$d(gy_n^2, p) = d((1- \beta_n^2)\, gx_n + \beta_n^2 fx_n^3, p)$$

$$\leq (1- \beta_n^2)\, d(\, gx_n, p) + \beta_n^2\, d(fx_n^3, p)$$

$$= (1- \beta_n^2)\, d(\, gx_n, p) + \beta_n^2\, d(fz, fx_n^3)$$

$$\leq (1- \beta_n^2)\, d(\, gx_n, p) + \beta_n^2\ \big[\delta\, d\big(gz, gy_n^3\big) + L(d(gz, fz))\big]$$

$$= (1- \beta_n^2)\, d(\, gx_n, p) + \beta_n^2\ \delta\, d\big(gz, gy_n^3\big). \tag{2.13}$$

Continuing above process, we have from (2.12) and (2.13), we get

$$d(gx_{n+1}, p) \leq (1- \alpha_n(1- \delta) - \delta\, \alpha_n\ \beta_n^1)\, d(gx_n, p)$$
$$+ \delta^2 \beta_n^1 \alpha_n\, (1- \beta_n^2)\, d(\, gx_n, p) + \delta^3 \alpha_n \beta_n^1 \beta_n^2\ \ d\big(p, gy_n^3\big)$$

$$= (1- \alpha_n(1- \delta) - (\, 1- \delta) \delta\, \alpha_n\ \beta_n^1) - \delta^2 \alpha_n \beta_n^1 \beta_n^2\ d(\, gx_n, p)$$

$$+ \delta^3 \alpha_n \beta_n^1 \beta_n^2\ \ d\big(p, gy_n^3\big) . \tag{2.14}$$

Similarly,

$$d(gy_n^3,\ p) \leq (1 - \beta_n^3)\, d(gx_n,\ p) + \delta\beta_n^3\, d(gy_n^4,\ p)\,. \qquad (2.15)$$

Continuing the above process, we have

$$\mathrm{d}\,(gx_{n+1},\ p) \leq (\,1 - \ (1\ \text{-}\ \delta\,)\,\alpha_n\ - \delta^{k-2}\,\alpha_n\ \beta_n^1\,\beta_n^2\beta_n^3.....\beta_n^{k-2})\ d(gx_n,\ p)$$

$$+\ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2} d(p,\ gy_n^{k-1})$$

$$\leq (\,1 - \ (1\ \text{-}\ \delta\,)\,\alpha_n\ - \delta^{k-2}\,\alpha_n\ \beta_n^1\,\beta_n^2\beta_n^3.....\beta_n^{k-2})\ d(gx_n,\ p)$$

$$+\ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}[(1 - \ \beta_n^{k-2})d\,(p,\ gx_n) + \beta_n^{k-1}\, d\,(fz,\ fx_n\,)]$$

$$\leq (\,1 - \ (1\ \text{-}\ \delta\,)\,\alpha_n\ - \delta^{k-2}\,\alpha_n\ \beta_n^1\,\beta_n^2\beta_n^3.....\beta_n^{k-2})\ d(gx_n,\ p)$$

$$+\ \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}[(1 - \ \beta_n^{k-2})d\,(p,\ gx_n) + \delta\beta_n^{k-1}\, d\,(p,\ gx_n\,)]$$

$$\leq (\,1 - \ (1\ \text{-}\ \delta\,)\,\alpha_n\ - \delta^{k-2}\,\alpha_n\ \beta_n^1\,\beta_n^2\beta_n^3.....\beta_n^{k-2})\ + (\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.......\beta_n^{k-2}\ d(\,gx_n\,,\ \ p)$$

$$\leq (\,1 - \ (1\ \text{-}\ \delta\,)\,\alpha_n)\ \mathrm{d}\,(gx_n,\ p)$$

$$\leq (\,1 - \ (1\ \text{-}\ \delta\,)\,)\ \mathrm{d}\,(gx_n,\ p)$$

$$\Rightarrow\ \mathrm{d}\,(gx_{n+1},\ p)\ \leq\ \ \mathrm{d}\,(gx_n,\ p)\,. \qquad (2.16)$$

Hence $gx_n \to p,$ since 1 - $\delta < 1$ for all n.

Now, we show that p is unique. Suppose there exist another point of coincidence is p^* , then there is an $z^* \in X$, such that $fz^* = gz^* = p^*$.

Now,

$$d(p,\ p^*) = d(fz,\ fz^*) \leq\ \ \delta\, d(gz,\ gz^*) + L\,(d(gz, fz)) =\ \delta d(gz,\ gz^*) = \delta\, d(p, p^*).$$

$$(1\ \text{-}\ \delta)d(p,\ p^*) \leq 0, \quad [\ \because\ \ (1\ \text{-}\ \delta) < 1]$$

$$\Rightarrow d(p,\ p^*) \leq 0\ \Rightarrow d(p,\ p^*) =\ 0.$$

So $p = p^*$ (i.e. p is unique.)

Since $fz = gz = p$, then $fgz = fp$ and $gfz = gp$ but $fgz = gfz$, so $fz = gz$. i.e. $fz = gz = p$ or $fp = gp = p$. So z is unique common fixed point of f *and* g.

Theorem 2.*3 Let (X, d) be a cone metric space and P is a cone in E. Consider two mappings f, g : X → E, which are weakly compatible at coincidence point p with f(X)⊆ g(X) and satisfy*

$$d(fx, fy) \leq h \max \left[\, d(gx,\, fx),\, d(gy,\, fx),\, d(gx,\, fy)\,,\; \frac{d(gx,\, fx) + d(gy,\, fy)}{2}, \frac{d(gx,\, fy) + \, d(gy,\, fy)}{2} \right], 0 \leq h < 1. \qquad (2.17)$$

Then Jungck multistep iteration $\{gx_n\}_{n=1}^{\infty}$ *converges to their coincidence point and f, g have a unique common fixed point of their coincidence point.*

Proof: Case-I : If

$$\begin{gathered} d(fx, fy) \leq h\, d(gx, fy) \\ \leq h\,[d\,(gx, fx) + d(fx, fy)] \\ (1-h)d(fx, fy) \,\leq h\, d\,(gx, fx) \\ d(fx, fy) \leq \frac{h}{1-h} d(gx, fx) \qquad (\because gx = fx) \\ \Rightarrow d(fx, fy) = 0 \;\Rightarrow fx = fy,\ \forall x, y\, \in X. \end{gathered}$$

Then $fx = gx = fy = gy$.

Now, we show that p is unique. Let p and p^* be two coincidence point of f and g such that $fx = gx = p$ and $fy = gy = p^*$, then

$$d(p,\, p^*) = d\,(fx, fy) \leq h\, d(gx, fy) = h\, d(p,\, p^*)$$

$$(1-h)d(p,\, p^*) \leq 0 \;\Rightarrow p = \, p^*\,.$$

Also, $fx = gx = p \implies gfx = gp$ and $gfx = fp$ but $fgx = gfx$, therefore $fp = gp \implies fp = gp = p$. Hence coincidence point is unique and therefore p is unique common fixed point of f and g.

Case-II: If

$$\begin{gathered} d(fx, fy) \leq h\, d(gy, fx) \\ \leq h[d(gy, fy) + d(fy, fx)] \\ d(fx, fy) \leq \frac{h}{1-h} d(gy, fy) \qquad (\because gy = fy) \\ \Rightarrow d(fx, fy) = 0 \;\Rightarrow fx = fy,\ \forall x, y\, \in X. \end{gathered}$$

So as in case first, p is unique common fixed point of f and g.

Case –III: If

$$\begin{gathered} d(fx, fy) \leq h\, d(gx, gy) \\ \leq h[d(gx, fx) + d(fx, gy)] \qquad (\because gx = fx) \end{gathered}$$

$$\implies d(fx, fy) \leq h\, d(fx, gy).$$

From case second, we have similar result.

Case-IV: If

$$d(fx, fy) \leq h \left[\frac{d(gx,\ fx) + d(gy,\ fy)}{2}\right].$$

Since point of coincidence is unique, so $fx = gx$ and $fy = gy$. Therefore proof is trivial.

Case-V: If

$$d(fx, fy) \leq h \left[\frac{d(gx,\ fy) + d(gy,\ fy)}{2}\right].$$

Since point of coincidence is unique, so $fy = gy$. Therefore

$$d(fx, fy) \leq h \left[\frac{d(gx,\ fy)}{2}\right]$$

So trivially satisfy as case first.

Hence in all cases, we have p is the unique common fixed point of f and g.

Example 2.1 Let $X = [0, a], a = 1, 2, 3$ and (X, d) be a cone metric space and mappings f and g such that $f(X) \subseteq g(X)$ and defined as

$$f(x) = \begin{cases} 0 : if\ x = 0 \\ \frac{x}{2} : if\ x \in (0, a] \end{cases}, \quad g(x) = \begin{cases} 0 : if\ x = 0 \\ x + a : if\ x \in (a-1, a] \end{cases}$$

Satisfy all the conditions in above theorems. It is clear that 0 is the coincidence point and common fixed point of f and g.

III. CONFLICT OF INTEREST:

The authors declare that there is no conflict of interest regarding the publication of this paper.

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